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INTEGRAL EQUATIONS OF SAINT-VENANT'S PROBLEM AND THE PROBLEM OF ANTIPLANE DEFORMATION†

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Non-classical integral equations for Laplace's equations, which give increased accuracy in numerical calculations, are employed to solve Saint-Venant's problem (the torsion and bending of a cylindrical rod by a force) and the problem of antiplane deformation. It is shown that for a unique determination of the solution of the initial problem in the case of multiply connected regions, the equations must be solved simultaneously with additional conditions, the number of which is determined by the connectedness of the region. The integral equations are solved analytically for certain specific regions: an infinite strip, a circle and a circular ring. © 2002 Elsevier Science Ltd. All rights reserved.

Two-dimensional problems of the theory of elasticity – torsion and antiplane deformation – were investigated in [1–3] by the method of boundary integral equations, where the harmonic functions of torsion or stress were investigated in the form of the potentials of a simple or double layer with unknown density.

The purpose of the present paper is to solve these problems, and also the problem of bending by a force, employing new integral equations for Laplace's equation, obtained previously in [4] by a direct method using an integral identity. In the two-dimensional case the kernels of these equations are identical with the kernels of the equations of harmonic potential theory. However, unlike the integral equations considered earlier in [1–3], the unknowns in these integral equations are boundary values of the stresses, which increases the accuracy with which they can be determined numerically. The right-hand sides of the integral equations are determined by the boundary conditions and the values of the divergence and curl of the vector of the shear stresses in the region, which are known in all three of the problems considered.

For multiply connected regions the solution of the integral equation is non-unique. It is shown that in this case the integral equation must be solved simultaneously with the condition for the axial displacement to be unique, expressed in terms of the circulation of the vector of the shear stresses.

Exact solutions of the integral equation are obtained for an infinite strip, a circle and a circular ring, which may serve as reliable tests of the use of the boundary-elements method.

1. DERIVATION OF THE INTEGRAL EQUATION

Consider a multiply connected region – the cross-section of a cylindrical body. We will use the following notation: G is an open region of connectedness m, $\Gamma = \Gamma_1 + \Gamma_2 + \Gamma_{m+1}$ is the Lyapunov boundary of the region, the contour Γ_i is the boundary of the ith opening – the region Gi ($i = 1, 2, \ldots, m$), Γ_{n+1} is the outer contour, $\mathbf{i}_\alpha$ ($\alpha = 1, 2$) is a Cartesian basis in the plane, $\mathbf{k} = \mathbf{i}_1 \times \mathbf{i}_2$ is the unit vector of the axis passing through the centre of inertia of the sections, O is the origin of the system of coordinates, $\mathbf{n}$ and $\mathbf{s} = \mathbf{k} \times \mathbf{n}$ are the outward normal and tangent to Γ, $\mathbf{r} = \mathbf{r}_p - \mathbf{r}_q$ is the radius vector from the source point q to the integration point p, and the prime denotes rotation of the vector by an angle $\pi/2$: $\mathbf{a}' = \mathbf{k} \times \mathbf{a}$.

The following identity holds for the plane vector $\boldsymbol{\tau} = \mathbf{i}_\alpha \tau_\alpha$ and the scaler θ

$$\int_G [\boldsymbol{\tau}\Delta\theta + (\nabla \circ \boldsymbol{\tau})\nabla\theta + (\nabla' \circ \boldsymbol{\tau})\nabla'\theta] dG = \int_\Gamma [\tau_n \nabla\theta + \tau_s \nabla'\theta] d\Gamma, \tag{1.1}$$

where $\nabla = \mathbf{i}_\alpha \partial/\partial x_\alpha$ is the two-dimensional Hamilton operator, $\Delta = \nabla \circ \nabla$ is the Laplace operator, and $\tau_n = \mathbf{n} \circ \boldsymbol{\tau}$ and $\tau_s = \mathbf{s} \circ \boldsymbol{\tau}$ are the normal and tangential components of $\boldsymbol{\tau}$.

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Suppose $\theta = (2\pi)^{-1} \ln r$ is the fundamental solution of Laplace's equation in the plane $r = |\mathbf{r}|$. Then, from identity (1.1) we obtain the following representation for the vector $\boldsymbol{\tau}$ at an arbitrary point q of the region G

$$\boldsymbol{\tau}(q) = \frac{1}{2\pi} \int_\Gamma \left(\tau_n \frac{\mathbf{r}}{r^2} + \tau_s \frac{\mathbf{r}}{r^2} \right) d\Gamma_p - \frac{1}{2\pi} \int_G \left(\frac{\mathbf{r}}{r^2} \nabla \circ \boldsymbol{\tau} + \frac{\mathbf{r}'}{r^2} \nabla' \circ \boldsymbol{\tau} \right) dG_p \tag{1.2}$$

Assuming that the component τ_s is continuous on Γ and τ_n satisfies the Lipschitz condition [5], by taking the limit $q \in G \to p_0 \in \Gamma$ in representation (1.2), projected onto the tangent $\mathbf{s}_0$ at the point p_0, we obtain an integral equation for τ_s

$$\tau_s(p_0) - \frac{1}{\pi} \int_\Gamma \tau_s(p) \frac{\mathbf{n}_0 \circ \mathbf{r}}{r^2} d\Gamma_p = \frac{1}{\pi} \int_\Gamma \tau_n(p) \frac{\mathbf{s}_0 \circ \mathbf{r}}{r^2} d\Gamma_p -$$

$$- \frac{1}{\pi} \int_G \left(\frac{\mathbf{s}_0 \circ \mathbf{r}}{r^2} \nabla \circ \boldsymbol{\tau} + \frac{\mathbf{n}_0 \circ \mathbf{n}}{r^2} \nabla' \circ \boldsymbol{\tau} \right) dG_p = f(p_0), \tag{1.3}$$

where $\mathbf{n}_0$ is the normal at the point p_0 and $f(p_0)$ denotes the right-hand side of the equation. The contour integral on the right-hand side of integral equation (1.3) must be understood in the sense of the Cauchy principal value [5].

2. DETERMINATION OF THE RIGHT-HAND SIDE OF THE INTEGRAL EQUATION FOR PROBLEMS OF THE THEORY OF ELASTICITY

Integral equation (1.3) can be used to solve the torsion problem (Problem 1), and the problem of bending by a force (Problem 2) and the second boundary-value problem of antiplane deformation (Problem 3), if the vector $\boldsymbol{\tau}$ is taken to mean the vector of the resultant shear stress.

The normal component occurring on the right-hand side of the integral equation satisfies the following equation

$$\tau_n|_\Gamma = T(s) \tag{2.1}$$

where $T(s) \equiv 0$ in Problems 1 and 2 and $T(s)$ is a known function of the arc coordinate – the specified external load in Problem 3.

Further, using well-known representations for the vector of the shear stresses τ in region G [6, 7], we calculate its divergence and curl

$$\nabla \circ \boldsymbol{\tau}(p) = -\mathbf{r}_p \circ \mathbf{I}^{-1} \circ \mathbf{Q}, \qquad \nabla' \circ \boldsymbol{\tau}(p) = 2\mu\alpha + \frac{\nu}{1+\nu} \mathbf{r}'_p \circ \mathbf{I}^{-1} \circ \mathbf{Q} \tag{2.2}$$

Here α is the torsion angle per unit length of the body axis in problem (the mean over the cross-section in Problem 2), $\alpha \equiv 0$ in Problem 3, $\mathbf{Q}$ is the transverse force ($\mathbf{Q} \equiv 0$ in Problems 1 and 3), $\mathbf{I} = \int_G \mathbf{r}_p \mathbf{r}_p dG_p$ is the tensor of the moments of inertia of the section, μ is the shear modulus and ν is Poisson's ratio.

Hence, the right-hand side in integral equation (1.3) is known in all three problems. It was obtained for Problem 1 previously† and can be represented in the form of a contour integral, which is convenient for practical calculations

$$f(p_0) = -\frac{2\mu\alpha}{\pi} \int_G \frac{\mathbf{n}_0 \circ \mathbf{r}}{r^2} dG_p = -\frac{2\mu\alpha}{\pi} \mathbf{n}_0 \circ \int_\Gamma \mathbf{n} \ln r d\Gamma_p = \frac{2\mu\alpha}{\pi} \int_\Gamma \frac{\mathbf{s}_0 \circ \mathbf{r} \mathbf{s} \circ \mathbf{r}}{r^2} d\Gamma_p$$

From the projections of τ_s obtained by solving integral equation (1.3) and the quantities τ_n, $\nabla \circ \boldsymbol{\tau}$, $\nabla' \circ \boldsymbol{\tau}$, known from (2.1)–(2.3), the shear stress T at any point of the region G is determined by integration in relation (1.2).

†NIKITIN, F. N., Boundary integral equations in stresses for problems of the statics of plane elastic systems. Candidate dissertation 05.23.17, Leningrad, 1990.

3. THE INTEGRAL EQUATION IN A MULTIPLY CONNECTED REGION; ADDITIONAL CONDITIONS WHICH ENSURE THAT THE INITIAL PROBLEMS OF THE THEORY OF ELASTICITY ARE UNIQUELY SOLVABLE

It is well known [5], that homogeneous integral equation (1.3) is conjugate to the homogeneous integral equation of the internal Dirichlet problem for Laplace's equation

$$\omega(p_0)+\frac{1}{\pi}\int_{\Gamma}\omega(p)\frac{\mathbf{n}\circ\mathbf{r}}{r^2}d\Gamma_p=0 \tag{3.1}$$

In the case of a single connected region, Eq. (3.1) has only a trivial solution, and in the case of a multiply connected region it has m linearly independent solutions [8]

$$\omega_i(p)=\begin{cases}C_i, & p\in\Gamma_i\\ 0, & p\notin\Gamma_i\end{cases}$$

Here and everywhere henceforth $i = 1, 2, \ldots, m$ and C_i are arbitrary constants. Then homogeneous integral equation (1.3) also has m linearly independent solutions. In problems 1–3 this is connected with the possible non-uniqueness of the axial displacement w in the multiply connected region.

Suppose b_i is the increment of the function w when going around the ith opening

$$b_i=\int_{\Gamma_i}\mathbf{s}\circ\nabla w d\Gamma \tag{3.2}$$

The condition $b_i = 0$ ensures that w is unique. When the expressions for ∇w are taken into account [6, 7], after calculations we obtain from (3.2) a generalized formula for the circulation of the shear stresses

$$\int_{\Gamma_i}\tau_s d\Gamma=\mu b_i-2\mu\alpha S_i-\frac{\nu}{1+\nu}\mathbf{Q}\circ\mathbf{I}^{-1}\circ\int_{G_i}\mathbf{r}_p' dG_p \tag{3.3}$$

where S_i is the area of the ith opening (region G_i).

Thus, for the multiply connected region integral equation (1.3) must be solved simultaneously with conditions (3.3).

4. EXACT SOLUTIONS OF THE INTEGRAL EQUATION FOR SOME OF THE SIMPLEST PROBLEMS

The torsion of an infinite strip $\{|x| < h/2, |y| < \infty\}$. The integrals over the unbounded region G and the boundary $\Gamma = \Gamma_+ \cup \Gamma_-$ are considered in the sense of the principal value. Then, the following system of integral equations follows from integral equation (1.3)

$$\tau_s^{\pm}(y_0)+\frac{h}{\pi}\int_{-\infty}^{+\infty}\frac{\tau_s^{\mp}(y)}{(y-y_0)^2+h^2}dy=2\mu\alpha h\quad(\tau_s^{\pm}(y)-\tau_s\,|\,\Gamma_{\pm}) \tag{4.1}$$

The solution of the system of integral equations (4.1) is a quantity $\tau_s^{\pm}(y) = \mu\alpha h$ which is constant on $\Gamma_{\pm}$. Using expression (1.2) we can determine the known shear stress [6] in the strip $\boldsymbol{\tau} = 2\mu\alpha x\mathbf{i}_2$.

Problems 1–3 for a circle. Integral equation (1.3) takes the following form in polar coordinates

$$\tau_s(\varphi_0)+\frac{1}{2\pi}\int_0^{2\pi}\tau_s(\varphi)d\varphi=f(\varphi_0) \tag{4.2}$$

Problems 1 and 2. The right-hand side of integral equation (4.2) has the form

$$f(\varphi_0)=2\mu\alpha a+\frac{1+2\nu}{4(1+\nu)I}a^2\mathbf{Q}\circ\mathbf{s}_0$$

where a is the radius and $I = \pi a^4/4$ is the moment of inertia of the circle. The solution of integral equation (4.2) is then

$$\tau_s(\varphi_0) = \mu\alpha a + \frac{1+2\nu}{4(1+\nu)I} a^2 \mathbf{Q} \circ \mathbf{s}_0$$

From formula (1.2) we determine the shear stress in the circle

$$\boldsymbol{\tau}(p) = \mu\alpha \mathbf{r}'_p + (8(1+\nu)I)^{-1}[((3+2\nu)a^2 - (1-2\nu)r_p^2)\mathbf{E} - 2(1+\nu)\mathbf{r}_p\mathbf{r}_p] \circ \mathbf{Q} \tag{4.3}$$

where $\mathbf{E} = \mathbf{i}_\alpha \mathbf{i}_\alpha$ is the unit tensor in the plane.

Problem 3. Suppose the external load has the following form: $T(\varphi) = \exp\,(in\,\varphi)$, $n \geqslant 1$. From the well-known properties of the singular Hilbert operator we have

$$f(\varphi_0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \exp(in\varphi)\operatorname{ctg}\frac{\varphi-\varphi_0}{2}d\varphi = i\exp(in\varphi_0)$$

The solution of integral equation (4.2) has the form $\tau_s(\varphi_0) = f(\varphi_0)$ and the shear stress in the circle will be

$$\boldsymbol{\tau}(p) = \exp(in\varphi_p)r_p^{n-2}a^{1-n}(\mathbf{r}_p + i\mathbf{r}'_p)$$

Problems 1 and 2 for a circular ring. Suppose a_k are the radii of the circles – the contours Γ_k; $\tau_s|\Gamma_k = \tau_{sk}, f|\Gamma_k = f_k$ $(k = 1, 2)$. For $p_0 \in \Gamma_k$ we can successively write the following integral equations in a polar system of coordinates

$$\tau_{sl}(\varphi_0) + \frac{(-1)^k}{2\pi}\int_0^{2\pi}\tau_{sl}(\varphi)d\varphi + \frac{a_{3k}}{\pi}\int_0^{2\pi}\frac{a_k - a_{3-k}\cos(\varphi-\varphi_0)}{r^2}\tau_{s,3-k}(\varphi)d\varphi = f_k(\varphi),\quad k=1,2 \tag{4.4}$$

The right-hand sides of system (4.4) have the form

$$f_k(p) = (k-1)H + F_k\mathbf{Q}\circ\mathbf{e}'_p,\quad \mathbf{e}_p = \frac{\mathbf{r}_p}{r_p},\quad H = 2\mu\alpha a_2\left(1 - \frac{a_1^2}{a_2^2}\right)$$

$$F_1 = -\frac{a_2^2 - a_1^2}{2(1+\nu)I},\quad F_2 = \frac{1+2\nu}{4(1+\nu)I}\left(1 - \frac{a_1^4}{a_2^4}\right)a_2^2;\quad I = \frac{\pi}{4}(a_2^4 - a_1^4)$$

(I is the moment of inertia of the circular ring).

Searching for a solution of integral equation (4.4) in the form

$$\tau_{sk}(\varphi_p) = C_k + A_k\mathbf{Q}\circ\mathbf{e}'_p + B_k Q\circ\mathbf{e}_p,\quad k=1,2$$

we obtain a subdefinite linear system of equations for the required constants

$$A_2 + \frac{a_1^2}{a_2^2}A_1 = F_2,\quad A_2 + A_1 = F_1,\quad B_2 + \frac{a_1^2}{a_2^2}B_1 = 0$$

$$B_2 + B_1 = 0,\quad C_2 + \frac{a_1}{a_2}C_1 = H \tag{4.5}$$

We will write the condition for the displacement (3.3) to be unique in the form

$$2\pi a_1 C_1 = -2\mu\alpha\pi a_1^2 \tag{4.6}$$

As a result, the solution of system (4.5) and (4.6) is

$$C_1 = -\mu\alpha a_1,\quad C_2 = \mu\alpha a_2,\quad B_1 = B_2 = 0$$

$$A_k = (-1)^k(4(1+\nu)I)^{-1}((1+2\nu)a_k^2 + (3+2\nu)a_{3-k}^2),\quad k=1,2$$

After fairly lengthy calculations, from (1.2) we determine the shear stress, written in coordinate-free form, not found in the literature. It differs from solution (4.3) by the replacement of a_2^2 by $a_1^2 + a_2^2 + a_1^2 a_2^2 r_p^{-2}$. In the special case when $\mathbf{Q} = \mathbf{Q}_1\mathbf{i}_1$ it is identical with the known formula [6], and it degenerates to solution (4.3) as $a_1 \to 0$.

5. THE INTEGRAL EQUATION FOR THE PROBLEM OF ANTIPLANE DEFORMATION FOR A DISPLACEMENT SPECIFIED ON THE BOUNDARY

The integral equation has a form similar to (1.3), namely

$$\begin{aligned}\tau_n(p_0) - \frac{1}{\pi}\int_\Gamma \tau_n(p)\frac{\mathbf{n}_0 \circ \mathbf{r}}{r^2}d\Gamma_p = -\frac{1}{\pi}\int_\Gamma \tau_s(p)\frac{\mathbf{s}_0 \circ \mathbf{r}}{r^2}d\Gamma_p - \\ -\frac{1}{\pi}\int_G\left(\frac{\mathbf{n}_0 \circ \mathbf{r}}{r^2}\nabla \circ \boldsymbol{\tau} - \frac{\mathbf{s}_0 \circ \mathbf{r}}{r^2}\nabla' \circ \boldsymbol{\tau}\right)dG_p\end{aligned} \tag{5.1}$$

The tangential component of the stress vector on Γ is known: $\tau_s = \mu \mathbf{s} \circ \nabla w|_\Gamma$. In the case of a simply connected region integral equation (5.1) is uniquely solvable, like integral equation (1.3). However, for a multiply connected region it solves an **m**-parametric family of boundary-value problems with displacements specified on the boundary, since, on the right-hand side, a tangential derivative of w occurs, which is not sensitive to mutual translations of the contours as rigid bodies. Hence, direct practical use of integral equation (5.1) is difficult in this case.

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